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CALCULATION OF TRANSPORT-EQUATION EIGENVALUES

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The eigenvalue problem for the transport equation with variable coefficients in an arbitrary domain with a smooth boundary is considered. A saturation-free numerical algorithm is constructed. Examples of numerical calculations are given, which prove the effectiveness of the proposed procedure.

Key words: *transport equation, filtration, eigenvalue problem.*

Introduction. Eigenvalue problems for the Laplace operator in an arbitrary smooth domain with constant coefficients were considered in [1]. Yet, many problems in mathematical physics require eigenvalue problems for a second-order equation with variable coefficients (see below) to be solved, which can be done by the steepest descent method [2, pp. 572 and 586]. This method, in particular, makes it possible to reduce the solution of a self-adjoint second-order equation to a sequence of solutions for the Poisson equation in the same domain, and can also be applied to nonlinear equations [3]. Nevertheless, the numerical examples considered in [3] show not too much optimism concerning the rapidity of convergence of the method. In the present study, a saturation-free numerical algorithm (see [4] for terminology) is constructed, suitable for solving the second-order elliptic equation with variable coefficients. As an example, the Neumann boundary condition is considered. In the course of the presentation of the method, we will show how other boundary conditions can be treated.

1. Formulation of the Problem on Gas Filtration in a Porous Medium. The sought solution has the form

$$\frac{\partial(m\rho)}{\partial t} + \operatorname{div}(\rho\boldsymbol{v}) = 0, \tag{1.1}$$

where $m = V_{\rm por}/V$ is the porosity (ranging in the interval of 0.15 to 0.22 for actual sedimentary beds), $m\rho$ is the concentration, and $\boldsymbol{v}$ is the filtration rate (not to be confused with fluid velocity!).

This equation can be obtained from the law of conservation of mass

$$\frac{d}{dt}\int\limits_{V_{\rm por}} \rho\, d\tau = \frac{d}{dt}\int\limits_{V} \rho m\, d\tau = 0, \tag{1.2}$$

where $V_{\rm por}$ is the volume of pores and V is the total volume, both volumes being variable quantities. We apply the formula of differentiation over a variable volume [5] to (1.2) and obtain (1.1).

The Darcy law holds for low-velocity fluid flows in an isotropic porous medium, i.e., for flows with low Reynolds numbers ($\mathrm{Re} < \mathrm{Re}_{\rm cr}$); this law can be expressed as

$$\boldsymbol{v} = -(\hat{k}/\mu)\operatorname{grad} p, \tag{1.3}$$

where $\hat{k}$ is the permeability coefficient measured in Darcy (1 D $= 10^{-8}/0.981$ cm^2) and μ is the dynamic viscosity. For actual porous media, $\hat{k} = 100$–1000 mD (1 mD $= 10^{-3}$ D). The permeability is a geometric characteristic of a porous medium, determined by the particle size, shape, and packing.

The equation of state is

$$\rho = \frac{M}{RT}\,\frac{p}{z(p)},$$

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where M is the molar weight of the gas, R is the universal gas constant, and T is the absolute temperature; the function $z(p)$ is to be determined experimentally [$z(p) = 1$ for a perfect gas], i.e., the gas is barotropic.

Equation (1.1) refers to the case of no gas sources (gas wells) present in the bed. Generally, the continuity equation is

$$\frac{\partial\,(m\rho)}{\partial t} + \operatorname{div}\,(\rho \boldsymbol{v}) = f(z,t) \qquad (z \in G), \tag{1.4}$$

where $f(z,t)$ is some given function and G is a two dimensional domain with a smooth boundary $\partial G \in C^\infty$. Let $z = \varphi(\zeta)$, where $\zeta = r\,\mathrm{e}^{i\theta}$ is the conformal mapping of a unit circle into the domain G. We write Eq. (1.4) in new variables [5, p. 180]:

$$ds^2 = (dr^2 + r^2\,d\theta^2)|\varphi'(\zeta)|^2 \quad \Rightarrow \quad g_{11} = |\varphi'(\zeta)|^2, \quad g_{22} = r^2|\varphi'(\zeta)|^2, \quad \sqrt{g} = |\varphi'(\zeta)|^2 r,$$

$$\operatorname{grad} p\Big|_r = \frac{1}{|\varphi'(\zeta)|}\,\frac{\partial p}{\partial r}, \qquad \operatorname{grad} p\Big|_\theta = \frac{1}{|\varphi'(\zeta)| r}\,\frac{\partial p}{\partial \theta}.$$

Then, we obtain

$$\frac{\partial\,(m\rho)}{\partial t} = |\varphi'(\zeta)|^{-2} L(w) + f(\zeta,t), \qquad \zeta = r\,\mathrm{e}^{i\theta}, \quad 0 \leqslant r \leqslant 1, \quad 0 \leqslant \theta < 2\pi, \quad |\zeta| \leqslant 1; \tag{1.5}$$

$$L(w) = \frac{1}{r}\,\frac{\partial}{\partial r}\Big(rk(r,\theta)\,\frac{\partial w}{\partial r}\Big) + \frac{1}{r^2}\,\frac{\partial}{\partial \theta}\Big(k(r,\theta)\,\frac{\partial w}{\partial \theta}\Big). \tag{1.6}$$

The following notation is used here: $m = m(r,\theta)$ is the porosity (a known function of coordinates), $p = p(r,\theta,t)$ is the pressure (an unknown function of coordinates and time), $\hat{k} = \hat{k}(r,\theta,p) = k(r,\theta)\psi(p)$ is the permeability (a known function of coordinates and pressure), $\rho = \rho(p)$ is the density (a known function of pressure), $\mu = \mu(p)$ is the viscosity (a known function of pressure), and $w(p) = \displaystyle\int \frac{\rho(p)\psi(p)}{\mu(p)}\,dp$.

The quantities m, θ, and ψ are dimensionless quantities, and the quantities p, ρ, and μ have the following dimensions: $[p] = M/(LT^2)$, $[\rho] = M/L^3$, $[\mu] = M/(LT)$, $[k] = L^2$, $[w] = M/(L^3T)$, and $[r] = L$. Here M is the mass unit, L is the length unit, and T is the time unit. The function $f(\zeta,t) = f(r,\theta,t)$ is the intensity of gas withdrawal, i.e., the mass of the gas released during a unit time in a unit volume of the bed. Introduction of the bed thickness $h = h(x,y)$ [i.e., bed height at the point $(x,y) \in G$ in the domain under consideration] does not change the form of (1.5), provided that m and k are substituted by mh and kh, respectively. In the latter case, we have $[f] = M/(L^2T)$, i.e., the value of f gives the mass of the gas released during a unit time from a unit surface area of the bed.

Thus, Eq. (1.5), together with (1.6), presents the sought formulation of the filtration problem. This equation needs to be supplemented by the boundary condition

$$\frac{\partial p}{\partial n}\Big|_{\partial G} = 0, \tag{1.7}$$

which implies that no gas flux penetrates through the boundary ∂G [see (1.3)]. Note that the function w also satisfies this boundary condition.

2. Discretization over Spatial Variables. For problem (1.5)–(1.7) to be discretized, we first perform discretization of the operator $L(w)$. Consider the spectral problem

$$L(w) + \lambda w = 0, \qquad \frac{\partial w}{\partial r}\Big|_{r=1} = 0. \tag{2.1}$$

Note that

$$-\int\limits_{|\zeta|\leqslant 1} L(w)w\,d\zeta = \int\limits_{|\zeta|\leqslant 1} \Big[k\Big(\frac{\partial w}{\partial r}\Big)^2 + \frac{k}{r^2}\Big(\frac{\partial w}{\partial \theta}\Big)^2\Big]\,d\zeta.$$

Thus, the boundary problem (2.1) is equivalent to the following extremum problem:

$$J(w) = \int\limits_{|\zeta|\leqslant 1} \Big[k\Big(\frac{\partial w}{\partial r}\Big)^2 + \frac{k}{r^2}\Big(\frac{\partial w}{\partial \theta}\Big)^2 - \lambda w^2\Big]\,d\zeta \to \min. \tag{2.2}$$

Indeed, δJ (variation of the functional J) is the principal linear part of the increment $J(w+h)-J(w)$, where h is an arbitrary smooth function. It is not difficult to show that

$$\delta J = 2 \int\limits_{|\zeta|\leqslant 1} \Big[k w_r h_r + \frac{k}{r^2}\, w_\theta h_\theta - \lambda w h\Big]\, d\zeta$$

$$= 2\Big\{ k r w_r h\Big|_{r=1} - \int\limits_{|\zeta|\leqslant 1} \Big[\frac{1}{r}\,\frac{\partial}{\partial r}\,(r k w_r) + \frac{1}{r^2}\,\frac{\partial}{\partial \theta}\,(k w_\theta) + \lambda w\Big] h\, d\zeta\Big\} = 0.$$

The function h is an arbitrary function; from here, relations (2.1) follow. To summarize, in searching for the minimum of functional (2.2), it is not necessary to satisfy the Neumann boundary condition beforehand, i.e., this boundary condition is natural.

To discretize functional (2.2), we use the quadrature formula [1]:

$$\int\limits_{|\zeta|\leqslant 1} f(\zeta)\, d\sigma = \sum_{\nu,l} c_{\nu l} f_{\nu l}, \qquad f_{\nu l} = f(r_\nu\, \mathrm{e}^{i\theta_l}),$$

$$r_\nu = \cos\frac{(2\nu-1)\pi}{4m}, \quad \nu = 1,2,\dots,m; \qquad \theta_l = \frac{2\pi l}{N}, \quad l = 0,1,\dots,2n; \quad N = 2n+1. \tag{2.3}$$

This quadrature formula can be obtained by substituting the integrand by the following interpolation formula for the function of two variables in a circle [4]:

$$(P_M f)(r,\theta) = \sum_{l=0}^{2n}\sum_{\nu=1}^{m} f_{\nu l} L_{\nu l}(r,\theta), \qquad f_{\nu l} = f(r_\nu, \theta_l),$$

$$L_{\nu l}(r,\theta) = \frac{2T_{2m}(r)}{NT'_{2m}(r_\nu)}\Big(\frac{D_n(\theta-\theta_l)}{r-r_\nu} - \frac{D_n(\theta-\theta_l+\pi)}{r+r_\nu}\Big), \tag{2.4}$$

$$D_n(\theta) = \frac{1}{2} + \sum_{k=1}^{n} \cos k\theta, \qquad T_m(r) = \cos\,(m \arccos r).$$

The interpolation formula (2.4) possesses all required properties. Indeed, this formula is an exact one on all polynomials of two variables of degree $\omega = \min\,(n, m-1)$. We denote the set of these polynomials as P_ω and the best approximation of the function $f \in C[D]$ (D is a unit circle) by a polynomial from P_ω as E_ω. This defines the projector

$$P_M\colon \qquad C[D] \to L^M, \qquad L^M = L(L_1, \dots, L_M).$$

In addition, the following classical inequality holds:

$$|f(r,\theta) - (P_M f)(r,\theta)| \leqslant (1 + |P_M|_\infty) E_\omega(f). \tag{2.5}$$

In (2.5), $|P_M|_\infty$ is the norm of the projector P_M. As in the one-dimensional case, inequality (2.5) shows that the corresponding interpolation formula has no saturation. The norm of the projector P_M satisfies the relation

$$|P_M|_\infty = O(\ln^2 M),$$

and this estimate is easy to refine. With some assumptions concerning the smoothness of the class of interpolated functions, we can estimate the rate of decay of the best approximation E_ω as $M \to \infty$ and the interpolation inaccuracy of (2.4).

Let

$$f(r,\theta) = (P_M f)(r,\theta) + \rho_M(r,\theta; f),$$

where $\rho_M(r,\theta;f)$ is the interpolation error of (2.4) (remainder of interpolation). Then, the following Babenko theorem holds [4].

Theorem 1. *Consider a class of functions $H_\infty^M(K;D) \subset C(D)$ that satisfy the conditions*

$$\Big|\frac{\partial^{k+l} f}{\partial x^k\,\partial y^l}\Big| \leqslant K, \qquad k+l \leqslant \mu,$$

in a circle D. Then, provided that $f \in H_\infty^M(K;D)$, we have

$$|\rho_M(\,\cdot\,;f)|_\infty \leqslant c_\mu K M^{-\mu/2}\log^2 M, \tag{2.6}$$

where c_μ is some constant that depends on μ.

Thus, inspection of (2.6) shows that, with an identical number M of interpolation nodes, the rate of decay of the interpolation inaccuracy of (2.4) increases with increasing μ, i.e., with increasing smoothness of the interpolated function f. This means that the interpolation formula has no saturation.

Based on (2.4), we can easily construct a quadrature formula for definite integrals, provided that the integration domain is a circle. Indeed, the substitution of the integrand by (2.4) yields the quadrature formula (2.3), where $d\sigma$ is an element of area, $c_{\nu l}$ are the weighting coefficients, and $\delta(f)$ is the error. For $c_{\nu l}$, we have

$$c_{\nu l} = \int\limits_D L_{\nu l}(r,\theta)\,d\sigma;$$

hence, these coefficients are independent of l. We introduce into the present consideration a block-diagonal matrix

$$C = \operatorname{diag}(c_1, c_2, \dots, c_m),$$

where c_ν $(\nu = 1,2,\dots,m)$ are some diagonal $N \times N$ matrices with identical diagonal numbers. For the inaccuracy of the quadrature formula, we have the following estimate [4]:

$$|\delta(f)| \leqslant 2\pi E_\omega(f).$$

Note that all $c_{\nu l}$ are positive if the total number of interpolation nodes is sufficiently large.

For the coefficients of the quadrature formula (2.3), we have the expression

$$c_\nu = \frac{4\pi r_\nu}{m(2n_\nu+1)}\Big(\frac{\cos\psi_\nu}{2} + \sum_{s=3(2)}^{m-1} t_s \cos s\psi_\nu\Big),$$

$$t_s = \frac{1}{1+(-1)^{(s-1)/2}s}, \qquad \psi_\nu = \frac{(2\nu-1)\pi}{4m}, \qquad s \geqslant 1 \text{ is an odd number,}$$

and

$$\Big(\frac{\partial w}{\partial r}\Big)_{\zeta=\zeta_{\nu l}} = \sum_{\mu,p} H_{\nu l,\mu p} w_{\mu p}, \qquad \Big(\frac{\partial w}{\partial \theta}\Big)_{\zeta=\zeta_{\nu l}} = \sum_{p=1}^{N} B_{lp} w_{\nu p}.$$

The matrices B and H can be obtained by differentiation of the interpolation formula (2.4):

$$B_{lp} = \frac{2}{N}\sum_{k=1}^{n} k \sin k\,\frac{2\pi(l-p)}{N}.$$

To obtain the matrix H, we differentiate (2.4) with respect to r. First, we introduce the designations

$$A_{\mu\nu}^{(1)} = \frac{d}{dr}\Big(\frac{T_{2m}(r)}{(r-r_\nu)T_{2m}'(r_\nu)}\Big)_{r=r_\mu} = \frac{1}{m}\sum_{s=1}^{2m-1}\frac{s\cos s\psi_\nu \sin s\psi_\mu}{\sin\psi_\mu},$$

$$A_{\mu\nu}^{(2)} = \frac{d}{dr}\Big(\frac{T_{2m}(r)}{(r+r_\nu)T_{2m}'(r_\nu)}\Big)_{r=r_\mu} = -\frac{1}{m}\sum_{s=1}^{2m-1}\frac{s(-1)^s\cos s\psi_\nu \sin s\psi_\mu}{\sin\psi_\mu}.$$

Then, we perform the differentiation of (2.4) with respect to r; this yields

$$\frac{du(r,\theta)}{dr}\Big|_{\substack{r=r_\mu\\ \theta=\theta_p}} = \sum_{\nu=1}^{m}\Big(A_{\mu\nu}^{(1)} u_{\nu p} - \frac{2}{N}\sum_{l=0}^{2n} A_{\mu\nu}^{(2)} D_n(\theta_p+\pi-\theta_l)u_{\nu l}\Big),$$

TABLE 1

Size	$\sqrt{\lambda_2}$	$\sqrt{\lambda_6}$	$\sqrt{\lambda_{11}}$	$\sqrt{\lambda_{16}}$
8×11	1.76	3.77	5.2	6.9
10×21	1.7751	3.72	5.16	6.5
15×31	1.776 235 57	3.7317	5.1770	6.4787

TABLE 2

Size	$\sqrt{\lambda_2} \cdot 10^{-7}$	$\sqrt{\lambda_6} \cdot 10^{-7}$	$\sqrt{\lambda_{11}} \cdot 10^{-7}$	$\sqrt{\lambda_{16}} \cdot 10^{-7}$	$\sqrt{\lambda_{21}} \cdot 10^{-6}$
8×11	2.8451	5.8268	9.2409	1.2075	1.4927
10×21	2.5447	6.5814	9.3812	1.3895	1.5927
15×31	2.4840	7.0164	10.0838	1.1905	1.4384
20×41	2.5558	7.1376	9.5192	1.0873	1.4266

where

$$D_n(\theta_p + \pi - \theta_l) = \frac{1}{2} + \sum_{k=1}^{n} (-1)^k \cos k(\theta_p - \theta_l) \quad \Rightarrow \quad H_{\mu p,\nu l} = A^{(1)}_{\mu\nu}\delta_{pl} - \frac{2}{N} A^{(2)}_{\mu\nu} D_n(\theta_p + \pi - \theta_l).$$

It can be easily seen that H is an h-matrix [1]; hence, this matrix can be represented as

$$H = \frac{2}{N} \sum_{k=0}^{n}{}' \Lambda_k \otimes h_k. \tag{2.7}$$

Here, the prime at the summation sign indicates that the term with $k = 0$ should be taken with the coefficient 1/2, Λ_k $(k = 0, 1, \dots, n)$ is a matrix of size $m \times m$, h_k $(k = 0, 1, \dots, n)$ is a matrix of size $N \times N$:

$$h_{kij} = \cos\left(2\pi k(i - j)/N\right), \qquad i, j = 1, 2, \dots, N$$

(symbol $\otimes$ denotes the Kronecker product of matrices). Here, the matrices Λ_k have the form

$$\Lambda_{k\mu\nu} = (-1)^{k+1} A^{(2)}_{\mu\nu} + A^{(1)}_{\mu\nu}.$$

Thus,

$$\Lambda_{2k,\mu\nu} = \frac{2}{m} \sum_{s=2(2)}^{2m-1} \frac{s \cos s\psi_\nu \sin s\psi_\mu}{\sin \psi_\mu}, \qquad \Lambda_{2k+1,\mu\nu} = \frac{2}{m} \sum_{s=1(2)}^{2m-1} \frac{s \cos s\psi_\nu \sin s\psi_\mu}{\sin \psi_\mu}.$$

Below, we denote these matrices as $\Lambda^{(k)}_{\mu\nu}$. We write (2.7) in more detail:

$$H_{\nu l,\mu p} = \frac{2}{N} \sum_{k=0}^{n}{}' \Lambda^{(k)}_{\nu\mu} \cos k \, \frac{2\pi(p - l)}{N}, \qquad H_{\nu l,\tilde{\mu}\tilde{l}} = \frac{2}{N} \sum_{q=0}^{n}{}' \Lambda^{(q)}_{\nu\tilde{\mu}} \cos q \, \frac{2\pi(l - \tilde{l})}{N}.$$

Using the quadrature formula (2.3), we transform functional (2.2) into the following quadratic form:

$$J(w) = \sum_{\nu,l} c_{\nu l} \Big[k_{\nu l} \Big(\frac{\partial w}{\partial r}\Big)^2_{\zeta=\zeta_{\nu l}} + \frac{k_{\nu l}}{r_\nu^2} \Big(\frac{\partial w}{\partial \theta}\Big)^2_{\zeta=\zeta_{\nu l}} + \lambda w_{\nu l}^2 \Big]. \tag{2.8}$$

Differentiating (2.8) with respect to $w_{\tilde{\mu}\tilde{l}}$, we obtain

$$\sum_{\mu,p} B_{\tilde{\mu}\tilde{l},\mu p} w_{\mu p} + \sum_{p=1}^{N} A^{\tilde{\mu}}_{\tilde{l}p} w_{\tilde{\mu}p} - \lambda c_{\tilde{\mu}} w_{\tilde{\mu}\tilde{l}} = 0,$$

where

$$B_{\tilde{\mu}\tilde{l},\mu p} = \frac{4}{N^2} \sum_{k=0}^{n}{}' \sum_{q=0}^{n}{}' \Big\{ \sum_{\nu=1}^{m} c_\nu \Lambda^{(k)}_{\nu\mu} \Lambda^{(q)}_{\nu\tilde{\mu}} \sum_{l=0}^{2n} k_{\nu l} \cos k \, \frac{2\pi(p - l)}{N} \cos q \, \frac{2\pi(l - \tilde{l})}{N} \Big\},$$

$$A^{\tilde{\mu}}_{\tilde{l}p} = \frac{4}{N^2} \frac{c_{\tilde{\mu}}}{r^2_{\tilde{\mu}}} \sum_{k=1}^{n} \sum_{q=1}^{n} kq \Big\{ \sum_{l=1}^{N} k_{\tilde{\mu}l} \sin k \, \frac{2\pi(l - p)}{N} \sin q \, \frac{2\pi(l - \tilde{l})}{N} \Big\}.$$

This is the sought discrete analog of the eigenvalue problem

$$\operatorname{div}(k \operatorname{grad} w) + \lambda w = 0, \qquad r < 1, \qquad \frac{\partial w}{\partial r}\Big|_{r=1} = 0.$$

The discretization error can be estimated as described in [1] (see also [6]).

3. Numerical Results. Problem (2.1) was numerically solved in a circle ($k = 1$) and in a disturbed circle (epitrochoid, $k = 1$, $k \neq 1$). For the circle, the eigenvalues $\sqrt{\lambda_i}$ ($i = 1, 2, \dots$) are known: they are zeros of the Bessel function derivative. A comparison of the values of $\sqrt{\lambda_i}$ calculated for the circle with exact values shows that they are identical to the forth digit after the decimal point even on a 3×7 grid. Yet, this accuracy is worse than that provided by the procedure described in [1] for the equation with constant coefficients. In addition, we performed calculations for the epitrochoid [$\varphi(\zeta) = \zeta(1 + \varepsilon\zeta^n)$, $\varepsilon = 0.0625$, $n = 12$]; for this epitrochoid, the calculated eigenvalues are summarized in Table (3.9) in [1]. The data obtained by the present procedure are listed in Table 1 (accurate to all signs coincident with those in [1]).

Yet another calculation run was performed for the same epitrochoid with the function

$$k(r, \theta) = k_0(0.1 + r^2)(\sin 12\theta + 1.1), \qquad k_0 = 10^{-13}/0.981 \text{ m}^2 = 0.1 \text{ D}.$$

The results are summarized in Table 2.

Thus, it can be concluded that the accuracy in calculating the eigenvalues by the present procedure is admissible, and the discretization over spatial variables is adequate.

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